HOMEWORK 4 SOLUTIONS

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Problem 7.1

(iii) $\mathbb{Z}^+$, because every positive integer is greater than or equal to itself.

(iv) $\varnothing$, because no positive integer is greater than or equal to every positive integer. In particular, n is not greater than or equal to $n+1$.

Problem 7.2

(iii) This is true. Let $m \in \mathbb{Z}^+$, and set $n = m$. Since $m \leq m$, the statement is satisfied.

(iv) True. Let $m = 1$. Then for all $n \in \mathbb{Z}^+$ we have $m \leq n$.

(v) True. Let $n \in \mathbb{Z}^+$ and set $m = n$. Since $n \leq n$, the statement is satisfied.

Problem 7.4

(vi) False. Suppose there did exist such a y. When $x = 0$, we have $xy = 0 \neq 1$.

(vii) True. Let $n \in \mathbb{Z}^+$. If n is even, then n is even or odd. If n is not even, then by definition n is odd, so it is even or odd. In either case the statement holds.

Problem 7.5

Let $n \in \mathbb{Z}$ and suppose there exists $q \in \mathbb{Z}$ such that $n = 2q+1$. Then we have

$$n^2 = (2q+1)^2 = 4q^2 + 4q + 1 = 2(2q^2 + 2q) + 1.$$

Since $2q^2 + 2q$ is an integer, we can set $p = 2q^2 + 2q$. Thus, there exists $p \in \mathbb{Z}$ such that $n^2 = 2p + 1$ as desired.

Problem 7.7

Let A, B, C, and D be sets. Let $(x, y) \in (A \times B) \cup (C \times D)$. Then $(x, y) \in A \times B$ or $(x, y) \in C \times D$. First, suppose $(x, y) \in A \times B$. Then $x \in A$ and $y \in B$, so we have $x \in A \cup C$ and $y \in B \cup D$. Therefore $(x, y) \in (A \cup C) \times (B \cup D)$.

Now suppose $(x, y) \in C \times D$. Then $x \in C$ and $y \in D$, so we have $x \in A \cup C$ and $y \in B \cup D$. Therefore $(x, y) \in (A \cup C) \times (B \cup D)$.

To see that these sets are not always equal, let $A = \{1\}$, $B = \{2\}$, $C = \{3\}$, and $D = \{4\}$. Then we have

$$A \times B = \{(1, 2)\} \text{ and } C \times D = \{(3, 4)\}$$

so that

$$(A \times B) \cup (C \times D) = \{(1,2),(3,4)\}.$$

However,

$$(A \cup C) \times (B \cup D) = \{(1,2),(1,4),(3,2),(3,4)\}.$$

Problem 10

(i) First we'll show $(a,\infty) \subseteq [b,\infty) \implies a \geq b$. Let $(a,\infty) \subseteq [b,\infty)$ and suppose toward a contradiction that $a < b$. Let $x = a + \frac{b-a}{2}$. Then $x > a$, so $x \in (a,\infty)$. Since $(a,\infty) \subseteq [b,\infty)$, this implies $x \in [b,\infty)$, so $x \geq b$. On the other hand,

$$\begin{aligned} x &= a + \frac{b-a}{2} \\ &= \frac{a}{2} + \frac{b}{2} \\ &< \frac{b}{2} + \frac{b}{2} \text{ since } a < b \\ &= b. \end{aligned}$$

Thus $x < b$, which contradicts the fact that $x \geq b$. Therefore we may conclude that $a \geq b$.

Now we'll show $a \geq b \implies (a,\infty) \subseteq [b,\infty)$. Let $x \in (a,\infty)$. Then $x > a \geq b$, so $x \geq b$. Thus $x \in [b,\infty)$, so we may conclude that $(a,\infty) \subseteq [b,\infty)$.

(ii) First we'll show $[a,\infty) \subseteq (b,\infty) \implies a > b$. Let $[a,\infty) \subseteq (b,\infty)$, and suppose toward a contradiction that $a \leq b$. Let $x = b$. Then $x \geq a$, so $x \in [a,\infty)$. Since $[a,\infty) \subseteq (b,\infty)$, this implies $x \in (b,\infty)$, which means $x > b$. However $x = b$, so this is impossible. Therefore $a > b$.

Now we'll show $a > b \implies [a,\infty) \subseteq (b,\infty)$. Let $x \in [a,\infty)$. By definition, $x \geq a > b$, so we have $x > b$. Thus, $x \in (b,\infty)$. Therefore $[a,\infty) \subseteq (b,\infty)$.

Problem 12

The statement "There is a greatest number in the set A" can be expressed in quantifiers as

$$\exists x \in A, \forall y \in A, x \geq y.$$

The negation of the statement is "There does not exist a greatest number in the set A." We can express this in quantifiers as

$$\forall x \in A, \exists y \in A, x < y.$$